The Chinese University of Hong Kong
Department of Mathematics
MMAT 5340 Homework 8
Please submit your assignment online on Blackboard
Due at 18:00 p.m. on Monday, 31th Mar, 2025

1. Consider a Markov chain $X = (X_n)_{n\geq 0}$ with a state space $S = \{1, 2, 3, 4\}$ and the transition matrix

$$A = \begin{bmatrix} 0.2 & 0.4 & 0 & 0.4 \\ 0.3 & 0 & 0.7 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.1 & 0.9 & 0 \end{bmatrix}.$$

Find the period $d(i)$ of each state, and which states are aperiodic?

Solution:

For $n \geq 1$, we deduce $P^n(1,1) > 0$, from the transition matrix, hence

$$R(1) = \{1, 2, 3, \cdots\}$$

It follows that $d(1) = 1$ and state 1 is aperiodic. Similarly, we use the transition matrix to show

$$R(2) = \{2, 3, 4, \cdots\}, \quad R(3) = \{1, 2, 3, \cdots\}, \quad R(4) = \{3, 4, 5, \cdots\}$$

Hence $d(2) = d(3) = d(4) = 1$ and state 2, 3 and 4 are aperiodic.

2. Let $(X_n)_{n\in\mathbb{N}_0}$ be a simple random walk. The state space S of $(X_n)_{n\in\mathbb{N}_0}$ is the set $\mathbb{Z}$ of all integers. We set $X_0 = 0$, and let the transition matrix P be defined by

$$P(i,j) = \begin{cases} p, & j = i+1 \\ 1-p, & j = i-1 \\ 0, & \text{otherwise} \end{cases}$$

for some constant $p \in (0, 1)$.

Find the period $d(i)$ of each state, and which states are aperiodic?

Solution: Note that for $i \in S = \mathbb{Z}$, we have

$$P^{2n}(i,i) > 0 \quad P^{2n+1}(i,i) = 0.$$

Hence for any $i \in S$ $R(i) = \{2, 4, 6, \cdots\}$ and $d(i) = 2$. Then no state is aperiodic.

3. Let $(X_n)_{n\in\mathbb{N}_0}$ be a simple random walk on $\mathbb{Z}^d$. The Markov chain with a transition matrix is given as follows:

$$P(x,y) = \begin{cases} \frac{1}{2d} & \text{if } \|x-y\|_1 = 1 \\ 0 & \text{otherwise,} \end{cases}$$

for any $x, y \in \mathbb{Z}^d$, where $\|x-y\|_1 := \sum_{i=1}^d |x_i - y_i|$.

First, assume that $d = 2$.

We divide the four directions into two groups, e.g. {up, right} and {left, down}. The Markov chain X could return to the origin 0 after $2n$ steps, so choose n from $2n$ for the location of each group as the number of up and right should be equal to the number of left and down. Finally, for each group and each $k \le n$, we choose k from n for both groups since the number of up (right) should be equal to the number of down (left). It follows that the return probability in $2n$ steps is given by

$$P^{2n}(0,0) = 4^{-2n}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}^2.$$

(a) Show that

$$P^{2n}(0,0) = 4^{-2n}\binom{2n}{n}^2.$$

Hint: Consider the coefficient of $(1+x)^n(1+x)^n = (1+x)^{2n}$ for each x^k, $k \in \{0, 1, \cdots, 2n\}$. Then use Multinomial Theorem to deduce that $\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}$.

(b) Deduce that the series $\sum_{n=1}^{\infty} P^{2n}(0,0)$ diverge, so that the random walk in dimension $d = 2$ is recurrent.

Hint: Use Stirling's formula: $n! \approx \sqrt{2\pi n}(\frac{n}{e})^n$ for large n.

Solution:

(a) Since

$$(1+x)^n = \sum_{i=0}^{n}\binom{n}{k}x^k$$

then

$$\sum_{j=0}^{2n}\binom{2n}{j}x^j = (1+x)^{2n} = ((1+x)^n)^2 = \left(\sum_{i=0}^{n}\binom{n}{k}x^k\right)^2$$

By abstracting the coefficients of the term x^n on both sides of above equation gives

$$\binom{2n}{n} = \sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k} = \sum_{k=0}^{n}(\binom{n}{k})^2.$$

Hence

$$P^{2n}(0,0) = 4^{-2n}\binom{2n}{n}^2.$$

(b)

$$\begin{aligned}\sum_{n=1}^{\infty} P^{2n}(0,0) &= \sum_{n=1}^{\infty} 4^{-2n}\binom{2n}{n}^2\\ &= \sum_{n=1}^{\infty} 4^{-2n}\left(\frac{(2n)!}{n!n!}\right)^2\\ &\approx \sum_{n=1}^{\infty}\frac{1}{n\pi}\\ &= +\infty\end{aligned}$$

Next, we assume that $d = 3$.

Let us accept that, by similar arguments as in the case $d = 2$, the return probability of the Markov chain in $2n$ steps is given by

$$P^{2n}(0,0) = 6^{-2n}\binom{2n}{n}\sum_{i+j+k=n}\binom{n}{i,j,k}^2,$$

where $\binom{n}{i,j,k} = \frac{n!}{i!j!k!}$.

(a) Show that

$$\sum_{i+j+k=n}\binom{n}{i,j,k} = 3^n.$$

Hint: We recall that, by Multinomial Theorem,

$$\sum_{k_1+k_2+\cdots+k_m=n}\binom{n}{k_1,k_2,\cdots,k_m} x_1^{k_1}x_2^{k_2}\cdots x_m^{k_m} = (x_1+x_2+\cdots+x_m)^n.$$

(b) Let us consider the Gamma function $\Gamma : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, which is defined by,

$$\Gamma(x+1) := \int_0^\infty t^x e^{-t}dt, \quad \forall x \geq 0.$$

We accept that, the second order derivative,

$$\Gamma'(x+1) = \int_0^\infty t^x e^{-t}\log t\ dt, \quad \Gamma''(x+1) = \int_0^\infty t^x e^{-t}(\log t)^2\ dt.$$

Show that the second order derivative of $x \longmapsto \log(\Gamma(x+1))$ is nonnegative, and deduce that the function $x \longmapsto \log(\Gamma(x+1))$ is convex.

Hint: For two functions $g(t) := \log(t)$, $h(t) \equiv 1$, we define the inner product by $\langle g,h\rangle := \int_0^\infty g(t)h(t)t^x e^{-t}dt$ and then apply the Cauchy-Schwarz inequality.

(c) Recall that

$$\Gamma(k+1) = k!, \quad \text{for all positive integer } k.$$

Deduce that if $i + j + k = n$, then

$$\binom{n}{i,j,k} \leq \binom{n}{n/3, n/3, n/3}.$$

Finally, use Stirling's formula to show for some constant C,

$$\binom{n}{i,j,k} \leq C\frac{3^n}{n}.$$

Hint: For the first inequality, use Jensen's inequality for the convex function $\ln(n!)$.

(d) Deduce that $\sum_{n=1}^{\infty} P^{2n}(0,0) < \infty$, so that the random walk in dimension $d = 3$ is transient.

Hint: First write

$$P^n(0,0) = \binom{2n}{n}(\frac{1}{2})^{2n} \sum_{i+j+k=n} \binom{n}{i,j,k}^2 (\frac{1}{3})^{2n}.$$

Then use results in (b) and (d), and find the upper bound for $\binom{2n}{n}$.

Solution: For $d = 3$.

(a) Since

$$(x_1 + x_2 + x_3)^n = \sum_{i+j+k=n} \binom{n}{i,j,k} x_1^i x_2^j x_3^k.$$

Picking $x_1 = x_2 = x_3 = 1$, we have

$$3^n = \sum_{i+j+k=n} \binom{n}{i,j,k}.$$

(b) Given the product

$$\langle g, h\rangle := \int_0^\infty g(t)h(t)t^x e^{-t} dt.$$

Let $g(t) = \log t$ and $h(t) \equiv 1$. By the Cauchy-Schwarz inequality

$$|\langle g, h\rangle|^2 \leq \langle g, g\rangle\langle h, h\rangle$$

we have

$$(\int_0^\infty t^x e^{-t} \log t \; dt)^2 \leq \int_0^\infty t^x e^{-t} (\log t)^2 \; dt \int_0^\infty t^x e^{-t} \; dt.$$

It follows that

$$(\Gamma'(x+1))^2 \leq (\Gamma''(x+1))^2 \Gamma(x+1)$$

Hence

$$(\log(\Gamma(x+1)))'' = \frac{\Gamma''(x+1)\Gamma(x+1) - (\Gamma'(x+1))^2}{\Gamma^2(x+1)} > 0$$

and the function $x \longmapsto \log(\Gamma(x+1))$ is convex.

(c) Let

$$\phi(x) = \log(\Gamma(x+1)),$$

which is convex. By Jensen's inequality

$$\phi(\frac{i+j+k}{3}) \leq \frac{\phi(i) + \phi(j) + \phi(k)}{3}.$$

Then

$$\exp(\phi(\frac{i+j+k}{3})) \leq \exp(\frac{\phi(i)+\phi(j)+\phi(k)}{3})$$

For $i+j+k=n$, we have

$$\Gamma(\frac{n}{3}+1) \leq (i!j!k!)^{\frac{1}{3}}$$

which is equivalent to

$$(\frac{n}{3}!)^3 \leq i!j!k!.$$

It follows that

$$\frac{n!}{i!j!k!} \leq \frac{n!}{\frac{n}{3}!\frac{n}{3}!\frac{n}{3}!}$$

where $\frac{n!}{i!j!k!} = \binom{n}{i,j,k}$ Applying Stirling's formula: $n! \approx \sqrt{2\pi n}(\frac{n}{e})^n$ to right hand side of above inequality yields

$$\frac{n!}{\frac{n}{3}!\frac{n}{3}!\frac{n}{3}!} \approx \frac{\sqrt{2\pi n}(\frac{n}{e})^n}{(\sqrt{2\pi(n/3)}(\frac{n/3}{e})^{n/3})^3} = \frac{3\sqrt{3}}{2\pi}\frac{3^n}{n}$$

and

$$\binom{n}{i,j,k} \leq C\frac{3^n}{n}$$

where $C = \frac{3\sqrt{3}}{2\pi}$.

(d)

$$\sum_{n=1}^{\infty} P^{2n}(0,0) = \sum_{n=1}^{\infty} 6^{-2n}\binom{2n}{n}\sum_{i+j+k=n}\binom{n}{i,j,k}^2$$
$$= \sum_{n=1}^{\infty} 2^{-2n}\binom{2n}{n}\sum_{i+j+k=n}3^{-2n}\binom{n}{i,j,k}^2$$

Applying

$$\binom{n}{i,j,k} \leq C\frac{3^n}{n}$$

to derive

$$\sum_{n=1}^{\infty} P^{2n}(0,0) \leq \sum_{n=1}^{\infty} C2^{-2n}\binom{2n}{n}\frac{3^{-n}}{n}\sum_{i+j+k=n}\binom{n}{i,j,k}$$

Using

$$3^n = \sum_{i+j+k=n}\binom{n}{i,j,k}$$

we have

$$\sum_{n=1}^{\infty} P^{2n}(0,0) \leq C\sum_{n=1}^{\infty} 2^{-2n}\binom{2n}{n}\frac{1}{n}$$

Applying Stirling's formula to $\binom{2n}{n}$ yields

$$\sum_{n=1}^{\infty} P^{2n}(0,0) \leq \frac{C}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{1}{n^{2/3}} < \infty.$$

Hence the random walk in dimension $d = 3$ is transient.